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NOTE

Note on the Two-Point Boundary Value Numerical Solution of the Orr–Sommerfeld Stability Equation

1. INTRODUCTION

In [1], a two-point boundary value numerical method is presented. This method was used to solve numerically an eigenvalue problem of linear boundary value ordinary differential equations (BVODEs). This eigenvalue problem arises from the exponential decay of stationary perturbation of Poiseuille flow, in a channel and an axi-symmetric pipe. In a more recent paper, by Bramley, Dieci, and Russel [4], several popular approaches for solving eigenvalues problems for linear BVODEs were presented. They used methods similar to the one in [1] to solve, numerically, the Orr–Sommerfeld stability equation and comparison of results were given, for different strategies of implementations. In this paper, we extended the method in [1] to compute, directly from the Orr–Sommerfeld equation, the transition curve, associated to the stability–instability mode. This transition curve is defined for all the values $(R, \alpha(R))$ for which $\mathrm{imag}(\lambda(R)) = 0$, where R is the Reynolds number of the basic flow, α is the real wave number, and λ is the real wave speed (see, e.g., [8, pp 80–82]).

The method used in this paper can be described in two steps; first, we set the ordinary differential equation as an equivalent first-order system of ordinary differential equations with conditions at two end-points. Second, we numerically solve the resulting system using the two-point boundary value solver code PASVA3. We carried out calculations for symmetric disturbances over a wide range of Reynolds numbers yielding accurate results, in a more efficient manner than, say, using spectral methods. The numerical results obtained for the eigenvalues are in complete agreement with previous calculations, e.g., Bramley, Dieci, and Russell [4], Orszag [6], and Zebib [10], among others; however, the results presented in this paper are concerned only with the computation of the transition curve. This transition curve cannot be computed using a spectral method, since the resulting eigenvalue problem depends now on the two parameters α and λ. Although some of the methods described in [4] compete, in accuracy and efficiency, with the method described in this paper, results, regarding the computation of the transition curve, were not presented in [4].

In [1] the superiority of the present method with respect to the spectral method of Orszag, used by Bramley and Dennis [3], was demonstrated. The two main difficulties reported in [3] with respect to the spectral method Orszag, where the computation of spurious eigenvalues and the loss of accuracy, especially for high Reynolds number computations. However, as reported in [2, 11], the spectral method, sometimes, can be modified to remove the spurious modes encountered in the numerical solution of the eigenvalue problem. There are two disadvantages for the method described in [1]; first, we cannot compute the eigenvalues and eigenfunctions at a critical Reynolds number, i.e., a Reynolds number for which the solution is not isolated (see [1], for more details). Second, the eigenvalues and eigenfunctions are computed one at a time. However, these difficulties reported in [1] are usually offset by the substantial increase in accuracy, and efficiency, obtained when one computes the eigenvalues and their eigenfunctions.

We would like to remark that the importance of this work, compared with previous ones, is related to the accurate computation of the transition curve, directly from the Orr–Sommerfeld stability equation.

2. FORMULATION

The unsteady stream function ψ, for a two-dimensional incompressible viscous motion between parallel planes, satisfies the equation

$$\frac{\partial \nabla^2\psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2\psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2\psi}{\partial y} = \frac{1}{R}\nabla^4\psi, \qquad (2.1a)$$

with boundary conditions at the planes

$$\psi = \psi^0_{\pm}, \quad \frac{\partial \psi}{\partial y} = 0, \quad \text{at } y = \pm 1, \qquad (2.1b)$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, R is the Reynolds number, and $\psi^0_{\pm}$ are two given constants.

To study instability due to infinitesimal disturbances, we seek solutions of (2.1a)–(2.1b) as a small perturbation of Poiseuille flow in the form

$$\psi(t, x, y) = y - \frac{1}{3}y^3 + \varepsilon\tilde{\psi}(t, x, y), \qquad (2.2)$$

where ε is small quantity (see [9]). Substituting (2.2) in (2.1a),

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and neglecting squares of ε, leads to a linear equation for $\tilde{\psi}$. When $\tilde{\psi}(t, x, y) = \varphi(y) \exp(i\alpha(x - \lambda t))$ this equation is the Orr–Sommerfeld stability equation

$$\frac{d^4\varphi}{dy^4} - 2\alpha^2 \frac{d^2\varphi}{dy^2} + \alpha^4\varphi - i\alpha R\left\{(1 - y^2 - \lambda)\left(\frac{d^2\varphi}{dy^2} - \alpha^2\varphi\right) + 2\varphi\right\} = 0, \tag{2.3a}$$

with boundary conditions,

$$\varphi(\pm 1) = \frac{d\varphi}{dy}(\pm 1) = 0, \tag{2.3b}$$

where α is the positive wave number and λ is the complex wave speed, then instability occurs when $\text{imag}(\lambda) > 0$. As pointed out by Grosch and Salwen [5], the solution of (2.3) which gives instability has φ even in y. These instabilities occur for Reynolds numbers above 5000 (see, e.g., [8]). Following Orszag [6], we define the most unstable mode for the solution corresponding to the eigenvalue λ_1, with the smallest modulus, for which $\text{imag}(\lambda_1) > 0$, where we have assumed that the perturbed stream function $\tilde{\psi}$ is expanded as a series of the form

$$\tilde{\psi}(t, x, y) = \sum_n \varphi_n(y) \exp(i\alpha(x - \lambda_n t)), \tag{2.4a}$$

$$|\text{real}(\lambda_n)| \leq |\text{real}(\lambda_{n+1})|. \tag{2.4b}$$

$$\text{imag}(\lambda_n) \leq 0 \quad \text{for } n \neq 1. \tag{2.4c}$$

In this paper we are concerned with computations corresponding to the most unstable mode of the Orr–Sommerfeld stability equation.

3. METHOD OF SOLUTION

In this section we describe the method to solve numerically the eigenvalue problem for the Orr–Sommerfeld stability equation (2.3). This method can be described as follows: first, the eigenvalue problem is transformed into an equivalent two-point boundary value first-order system of the form

$$\vec{\mathbf{Z}}' = \mathbf{A}(\lambda, R, \alpha, y)\vec{\mathbf{Z}}, \quad a \leq y \leq b, \tag{3.1a}$$

$$\lambda' = 0, \tag{3.1b}$$

with boundary conditions

$$\mathbf{B}_a\vec{\mathbf{Z}}(a) + \mathbf{B}_b\vec{\mathbf{Z}}(b) = 0, \tag{3.1c}$$

$$z_k(b) = 1, \tag{3.1d}$$

for some k, $1 \leq k \leq N$. Here $\vec{\mathbf{Z}} = (z_1, z_2, \cdots, z_N)$, and $\mathbf{A}$, $\mathbf{B}_a$, $\mathbf{B}_b$ are $N \times N$ matrices. The prime in (3.1a) indicates derivatives with respect to y. The condition (3.1d) has to be chosen in a way that is not in conflict with (3.1c). In the case of Eq. (2.3) we used for condition (3.1d) $\varphi''(1) = 1$. To numerically solve the system (3.1) we used the two-point boundary value system code DVCPR from the IMSL library (PASVA3 [7]); it is a standard solver for first-order systems of ordinary differential equations with conditions at two end-points. It uses a variable step, with an automatic criterion to select a non-uniform grid and a variable order of accuracy, with an excellent correspondence between the requested tolerance and the actual global error in the numerical solution (see [7]). Since the systems considered are non-linear we used as an initial iterate in-PASVA3 the solution to an eigenvalue problem which is closely related to the Orr–Sommerfeld equation, which is known, and for which the boundary conditions are satisfied, to generate solutions for an arbitrary Reynolds number; this procedure is called successive continuation (see [1]). Using this method the eigenvalues and eigenfunctions are jointly computed, the computation of spurious eigenvalues is avoided, and for high Reynolds numbers the solution can be computed in a very accurate manner.

4. INITIAL ITERATE

Since the system (3.1a)–(3.1d) is nonlinear in $(\vec{\mathbf{Z}}, \lambda)$ we used as an initial iterate in PASVA3 solutions of the problem

$$\frac{d^4\varphi}{dy^4} - 2\alpha^2 \frac{d^2\varphi}{dy^2} + \alpha^4\varphi = \lambda^4\varphi, \tag{4.1a}$$

with boundary conditions

$$\varphi(\pm 1) = \varphi'(\pm 1) = 0. \tag{4.1b}$$

This equation has even and odd exact solutions; the even solutions are given by

$$\varphi(y) = \bar{C}\left[\frac{\cosh(\beta y)}{\cosh(\beta)} - \frac{\cos(\gamma y)}{\cos(\gamma)}\right], \tag{4.2a}$$

$$\beta = (\lambda^2 + \alpha^2)^{1/2}, \quad \gamma = (\lambda^2 - \alpha^2)^{1/2}. \tag{4.2b}$$

In order to satisfy (4.1b) λ has to be a solution of the transcendental equation

$$\beta \tanh(\beta) + \gamma \tan(\gamma) = 0. \tag{4.2c}$$

Finally, $\bar{C}$ is a normalizing constant such that

$$\varphi''(1) = 1. \tag{4.2d}$$

Grosch and Salwen [6] used solutions of (4.2) to construct a series expansion approximation to φ in (2.3a). We used as initial iterate, for PASVA 3, the solution (φ, λ) of (4.2a)–

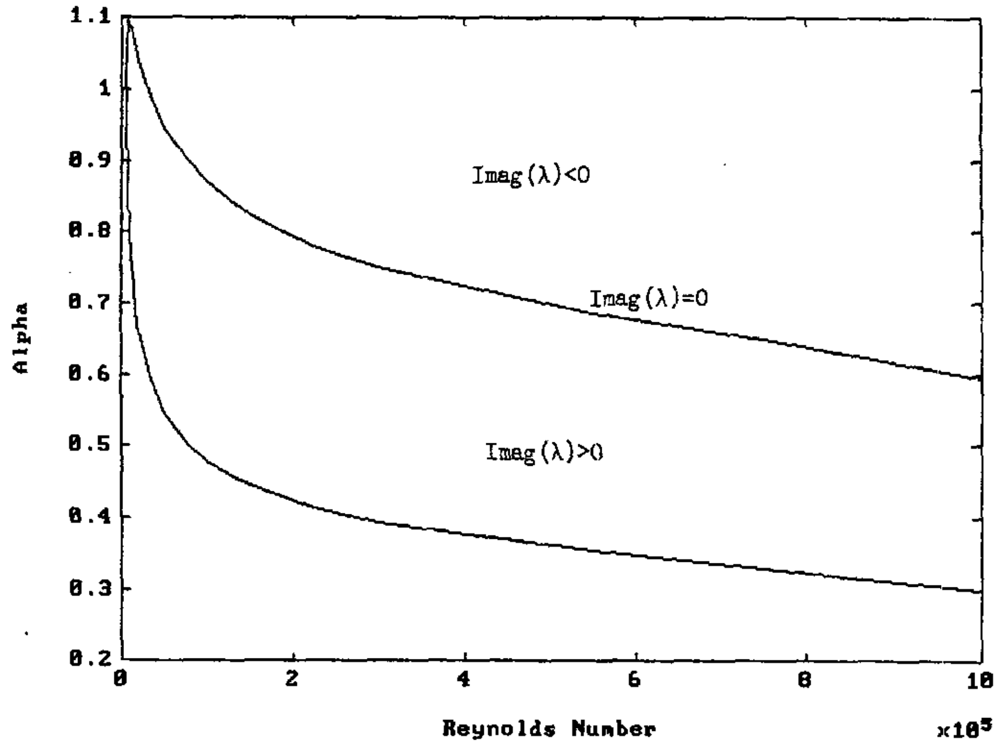

FIG. 1. Transition curve for $R \in [5772.22, 10^6]$.

(4.2d); only three or four iterations were needed by the program to compute an accurate solution for each Reynolds number. The nonlinear equation (4.2c) was solved using the Newton method with complex arithmetic. Using the approximation of $\lambda_1 = (3\pi/2, 0)$ for the Newton method (see [6]) we obtained the value of $\lambda_1 = (2.48946, 0)$, which corresponds to the eigenvalue with smallest modulus in Eq. (4.2c).

5. NUMERICAL RESULTS

Although the results presented in this paper are concerned with the computation of the transition curve, we have made computations which were in complete agreement with the previous results presented in [4, 6, 8], concerning the computation of the complex speed λ. All the computations were done on the IBM 3081 at the IBM Research Center, Caracas-Venezuela, and on the VAX 11/780 at the Center for the Mathematical Sciences, University of Wisconsin–Madison. The program PASVA3 was used on the IBM 3081 while the program DVCPR, which is a PASVA3 version from the IMSL library, was used on the VAX 11/780.

The way we present these results is the following: for a particular wave number $\alpha_c = 1.0256$ (the critical wave number), using the initial iterate of Section 4 on PASVA 3, we compute the solution corresponding to Reynolds number $R = 1$. Applying successive continuation, on PASVA3, with $R = 1$, 10, 50, 100, 200, 500, 1000, 2000, we compute the solution at $R = 5772.22$. This last Reynolds number is the critical Reynolds number (R_c), i.e., the smallest Reynolds number for which imag(λ) = 0. At $R = R_c$, we obtained the value of $\lambda = 0.264001730 + i0.3696 \times 10^{-9}$, which agrees with the value obtained by Orszag, up to eight decimal places (see [6]). Second, we compute the values $(\alpha(R), \lambda(R))$ with imag($\lambda(R)$) = 0, by solving numerically the system of Eqs. (3.1a), (3.1b), and

$$\alpha' = 0, \tag{5.1}$$

and the boundary conditions (3.1c), (3.1d). This formulation is possible since λ and α are considered, on the program, as real variables, while the conditions (3.1c) and (3.1d) are considered, on the program, as complex conditions, making the two-point boundary value problem well defined. Then we plot all the computed pairs $(R,\alpha(R))$ in the plane R,α which determine the transition curve, as a double valued-function on R (see Fig. 1).

The upper branch of the transition curve was computed using as an initial iterate for PASVA3 the numerical solution $(\vec{\mathbf{Z}}(y),\lambda)$, corresponding to α_c and R_c, and using as an initial iterate for α, $\alpha = \alpha_c + \delta$, with δ a small quantity (we used $\delta = 0.01$); then we used successive continuation. The lower branch of the transition curve was computed in a similar way, but using as the initial iterate for α, $\alpha = \alpha_c - \delta$. The computation of each branch was made separately starting from $R = 1$, using successive continuation, switching λ from a complex variable to a real variable at $R = 5772.22$ (the critical Reynolds number), adding Eq. (5.1), and using successive continuation from $R = 5772.22$ to $R = 10^6$. The CPU time to compute, for example, the entire upper branch was 5 min 24 s on the IBM 3081.

6. CONCLUSION

The numerical results we presented in this paper are concerned with the computation of the transition curve of the Orr–Sommerfeld stability equation. However, we have obtained results, regarding the computation of the complex or real speed λ, that were in agreement with the previous results [4–6, 10]. The procedure used to compute the eigenvalues and the eigenfunctions is based upon expressing the Orr–Sommerfeld equation as a two-point boundary value system. The eigenvalues were computed one at a time, jointly with the eigenfunctions, using successive continuation on the code PASVA 3. We think that our procedure is simple, direct, and very efficient and that it can be used with relative ease on a computer.

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